The Concept of Limit in Ancient Greek Mathematics

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Although the definition of limit was initially constructed by Augustin-Louis Cauchy and later amended by various mathematicians such as Karl Weierstrass, attempts at understanding and utilizing this concept had already begun in ancient Greece. In particular, mathematicians had started pondering the concept of convergence and limit in a axiomatic, quantitative fashion and utilize convergent sequences and series in problems on geometry.

The attempt to understand the concept of limit can be traced back to Zeno of Elea, a Greek philosopher and mathematician, and the two paradoxes he brought up [1]. The first paradox, the Dichotomy Paradox, states that if a person wants to walk to a wall, they must first reach the halfway point. After that, the remaining distance, which is half the original, still has to be covered, thus requiring the person to reach the halfway point of that, yet a quarter of the whole distance remains. This recursive process could be repeated infinitely, so it seems impossible for the person to reach the wall, for the process is never-ending. In the second paradox, the Achilles and the Tortoise Paradox, Achilles tries to chase a tortoise from behind. Denote the beginning position of the tortoise x_0. Before Achilles passes the tortoise, he must first reach x_0. But by the time Achilles does so, the tortoise has moved forward to a different position, denoted x_1. Achilles then has to move to x_1, but by then the tortoise would have moved to x_2, and this process goes on infinitely as well. Achilles seemingly could never catch up with the tortoise. It is both a mathematical intuition and real-life common sense that a person is able reach a wall and that a person could catch up with a tortoise. What contradiction induced this paradox?

The Dichotomy Paradox and the Achilles-Tortoise Paradox are constructed in the same fashion. We could see this using an anachronistic tool in physics unavailable to the ancient Greek mathematicians. Suppose in the second example, we could setup the reference frame such that the tortoise is immobile. Then the tortoise essentially equivalent to the wall in the first paradox, and Achilles (the person) would be moving towards the tortoise (the wall), with the magnitude of his speed equal to the difference between their speed relative to the ground. Then without loss of generality, let us focus our attention on the first paradox, the resolution of which implies the resolution of the second paradox as well.

Aristotle's answer to the Zeno's Paradox is that quantitative infinity and infinity of divisibility are two different concepts [2]. A value can be finite in magnitude, yet infinitely divisible into smaller portions. That is to say, from a bottom-up perspective, that an infinite series of numbers could sum up to a finite value. Aristotle also recognizes the shared nature of the Dichotomy Paradox and the Achilles Paradox and claims that "the solution must be the same" [2].

Archimedes, on the other hand, used a quantitative approach similar to how modern mathematicians would do, minus the rigorousness of defining the concept of limit. He derived the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...=1$ [3]. We now know to write the infinite series in terms of the limit $\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{2k}=1$ instead, then to evaluate it by Cauchy and Weierstrass' $\epsilon-N$ and $\epsilon-\delta$ definition of limit. This rigorous approach, of course, was invented centuries later [4], and Archimedes' approach is exceptional for his time.

The implication of Archimedes' approach is much greater, as solving complicated geometric problems becomes possible under the same method. Archimedes solved two geometric problems in this way: the Quadrature of the Parabola, that is, to calculate the area between a parabola and a line, and to calculate π, the ratio between the circumference and the diameter of a circle.

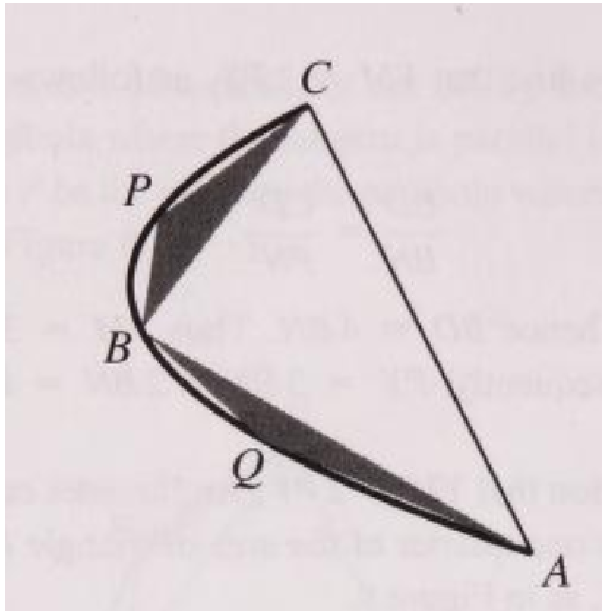

Figure 1: Triangles in the Enclosed Area [5]

Archimedes' approach to calculating the area bounded by the parabola and the line AC in Figure 1 is as follows. First, he would construct a point B such that the tangent line of the parabola at point B is parallel to line AC; points P and Q are then constructed in the same fashion with respect to line BC and line AB. He then concluded that the areas of the smaller triangles are each $\frac{1}{8}$ of the larger triangle. If we denote the area of triangle ABC as a, then the sum of the area of the two shaded triangles would be $a \cdot \frac{1}{8} \cdot 2 = \frac{1}{4}a$. Since P and Q are constructed in the same way as point B, we can apply the same procedure on the triangles PBC and QAB as well. This would result in 4 smaller triangles whose areas' sum is $\frac{1}{4}$ the sum of PBC and QAB's area, or $\frac{1}{4}a \cdot \frac{1}{4} = \frac{1}{16}a$. If we repeat this process, we would obtain the series $a + \frac{1}{4}a + \frac{1}{16}a + ... + \frac{1}{4n}a$, or $a \sum_{k=1}^{n} 4^{k-1}$, whose value represent the sum of the areas of all the triangles and, as Archimedes noticed, approaches the enclosed area as n approaches infinity.

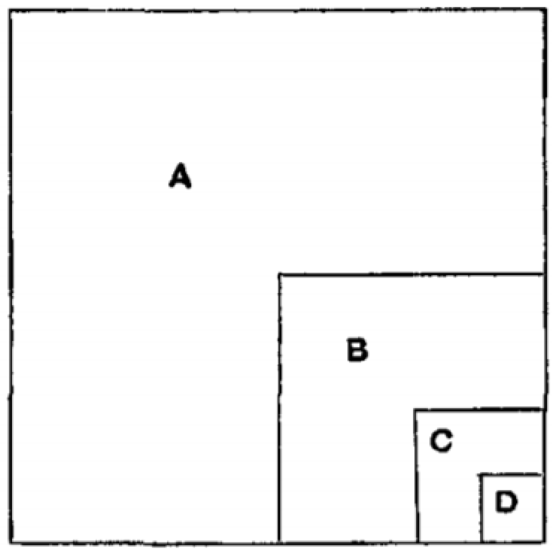

Figure 2: Series Visualized [6]

Our burden now shifts to solving this infinite series. Archimedes set up a series of square areas, represented by Figure 2, for which $A = 4B$, $B = 4C$, etc. Then $\frac{4}{3}B + \frac{4}{3}C + \frac{4}{3}D = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$. By moving terms, we derive $A + B + C + D + \frac{1}{3}D = \frac{4}{3}A$. If we continue this partition, the area of the smallest square approaches zero, so the last term of the left hand side vanishes. If we let $A = 1$, we would obtain the series $1+\frac{1}{4}+...+(\frac{1}{4})^n = \frac{1-(\frac{1}{4})^{n+1}}{1-\frac{1}{4}}$ in modern notation. At this point, Archimedes used an argument somewhat similar to the modern $\epsilon - \delta$ approach, and demonstrated that the infinite sum could be neither greater nor less than $\frac{4}{3}$ [7]. The quadrature problem is essentially solved at this point.

Archimedes' approach to calculating the circumference of the circle is to create a pair of inscribed and circumscribed regular polygons to give a tight bound for the circumference of the circle. Suppose the pair of regular polygon has n sides. It follows that $\theta = \frac{2\pi}{n}$, and by definition we see that the side length of the inscribing n-gon is $2\sin(\frac{1}{2}\theta)$ and the side length of the circumscribing n-gon is $2\tan(\frac{1}{2}\theta)$, and we obtain the circumference after multiplying each by n. Unfortunately, during Archimedes' time, which was three centuries BCE, even the most rudimentary form of trigonometry had

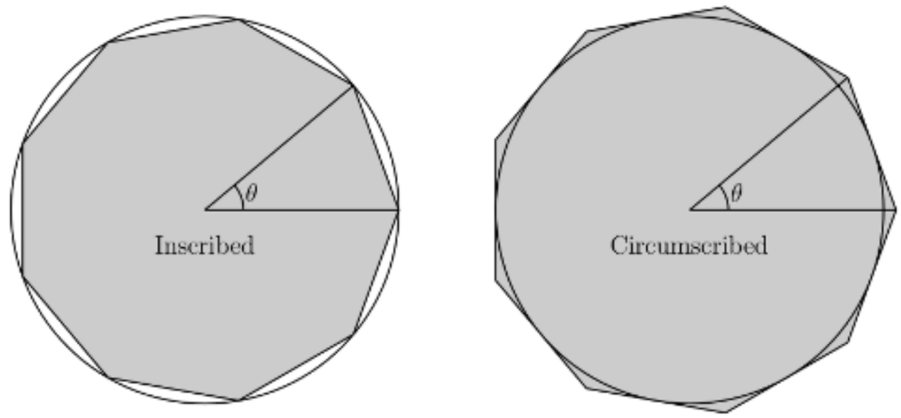

Figure 3: Side Lengths of inscribed and circumscribed regular polygon [8]

yet to be invented, so an analysis of this pair of limit is unrealistic. The actual proof by Archimedes was slightly more arduous.

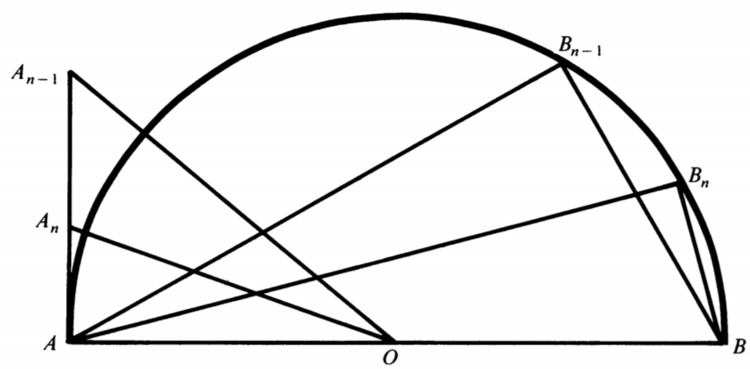

Figure 4: Archimedes' geometrical approach [8]

First, Archimedes would take a (2^nM)-gon, a subsequence of natural number n-gons, for ease of calculation (M is an integer greater than 3). It is possible for us to develop an implicit relationship between the n-th term and the $n-1$-th term of the sequence. Take the point A_n, a vertex of the circumscribing $M2^n$-gon, and A the midpoint of a side adjacent to A_n. By definition of circumscription, A lies on the circle. Then, take A_{n-1} such that A, A_n, and A_{n-1} are on the same line and that $2\angle AOA_n = \angle AOA_{n-1}$. Point

A_{n-1} would then be a vertex of the circumscribing $M2^{n-1}$-gon. Then we'd have

$$\begin{aligned}\frac{OA_{n-1}}{OA} &= \frac{A_{n-1}A_n}{AA_n}\\ \frac{OA+OA_{n-1}}{OA} &= \frac{AA_n + A_{n-1}A_n}{AA_n} = \frac{AA_{n-1}}{AA_n}\\ \frac{OA}{AA_n} &= \frac{OA}{AA_{n-1}} + \frac{OA_{n-1}}{AA_{n-1}}\end{aligned}$$

What Archimedes did not know, was that this equation translates to a relationship between trigonometric values: if we let a side of the $M2^n$-gon correspond to the angle $\theta = \frac{2\pi}{M2^n}$ at the center of the circle, we would have $\angle AOA_n = \frac{1}{2}\theta$, and in modern notation, $\frac{OA}{AA_n} = \cot(\frac{1}{2}\theta)$ and $\frac{OA_n}{AA_n} = \csc(\frac{1}{2}\theta)$. The equation above essentially becomes $\cot(\frac{1}{2}\theta) = \cot(\theta) + \csc(\theta)$.

On the other hand, we construct B_n and B_{n-1} on the other side of the circle such that $\angle BAB_n = \angle AOA_n$ and $\angle BAB_{n-1} = \angle AOA_{n-1}$. We'd have $\angle BOB_n = \theta$ and $\angle BOB_{n-1} = 2\theta$, making B_nB a side of the inscribing $M2^n$-gon and $B_{n-1}B$ a side of the inscribing $M2^{n-1}$-gon. This creates the sequence $P_{2^nM} = \frac{2^nM\cdot AB}{\cot(\frac{2\pi}{M2^n})}$ and $p_{2^nM} = \frac{2^nM\cdot AB}{\csc(\frac{2\pi}{M2^n})}$ where P is the circumference of the circumscribing polygon and p the inscribing. Of course, Archimedes wouldn't write "cot" or "csc"; he would write it in terms of a fraction of side lengths instead. The in the same nature as the Squeeze Theorem, these 2 sequences would eventually monotonically converge to the value of π from both directions. Archimedes took $M = 6$ and $n = 4$ to calculate a bound of π [9]. Unfortunately, the Ancient Greek lacked a positional arithmetic system, which hindered the efficiency of their numerical calculation. Vertical addition or multiplication was impossible under their number system, and abacus was used instead. While the calculation of Archimedes is limited in magnitude, the algorithm clearly reflected the methodology of using sequences to approach or bound a certain limit.

These works in ancient Greece demonstrated that mathematicians back in their era were already dealing with the concept and application of limit. Though their studies are restricted by the absence of rigorous definition and limitation of numerical calculation, ancient Greek mathematicians prompted their successors centuries later to utilize limit in problems and inspired them to develop the modern definition as we know today, leading to the invention of calculus, measure theory, and much more.

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